# Note not for publication: “An Analysis of Portion Cap Rules with a Multi-Product Seller”

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## 1 Model with related goods

In this note, I explore the more general case of related goods (products that are complement or substitute in consumption). The most important modification to the model is related to the taste parameters. For complements (superscript $C$), the following inequalities hold: ($\theta^C_{HH} > \theta^A_H + \theta^B_H$), ($\theta^C_{HL} > \theta^A_H + \theta^B_L$), ($\theta^C_{LH} > \theta^A_L + \theta^B_H$), and ($\theta^C_{LL} > \theta^A_L + \theta^B_L$). Additionally, $\theta^C_{HH} > \theta^C_{HL}$, $\theta^C_{HH} > \theta^C_{LH}$, $\theta^C_{HH} > \theta^C_{LL}$, $\theta^C_{HL} > \theta^C_{LL}$, and $\theta^C_{LH} > \theta^C_{LL}$. For substitutes (superscript $S$), the following holds: ($\theta^S_{HH} < \theta^A_H + \theta^B_H$), ($\theta^S_{HL} < \theta^A_H + \theta^B_L$), ($\theta^S_{LH} < \theta^A_L + \theta^B_H$), and ($\theta^S_{LL} < \theta^A_L + \theta^B_L$). Additionally, $\theta^S_{HH} > \theta^S_{HL}$, $\theta^S_{HH} > \theta^S_{LH}$, $\theta^S_{HH} > \theta^S_{LL}$, $\theta^S_{HL} > \theta^S_{LL}$, and $\theta^S_{LH} > \theta^S_{LL}$.

Because the analysis below is identical for complements and substitutes, I drop the superscript. Throughout the text, I use complements and related goods as interchangeable terms. The analysis and conclusions hold without change for substitute products. The seller's expected profit is the

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following:

$$E[\pi] = \underbrace{\sum_{ij} \beta_{ij}[\theta_{ij}u(q_{ij}^A) + \theta_{ij}u(q_{ij}^B) - c(q_{ij}^A) - c(q_{ij}^B)]}_{\text{Expected total surplus}} - \underbrace{\sum_{ij} \beta_{ij}[\theta_{ij}u(q_{ij}^A) + \theta_{ij}u(q_{ij}^B) - p_{ij}]}_{\text{Expected consumer surplus}} \tag{1}$$

The general form of the PC constraints remains PC: $R_{ij} \geq 0 \; \forall \; ij$. The IC constraints take the following form:

$$\text{IC: } R_{ij} \geq R_{kl} + \bar{u} + u(q_{kl}^A)(\theta_{ij} - \theta_{kl}) + u(q_{kl}^B)(\theta_{ij} - \theta_{kl}) \; \forall \; ij \text{ and } kl; i \neq k \text{ and } j \neq l \tag{2}$$

The following definitions will be useful:

$$\begin{aligned} &\theta_{HH} - \theta_{LL} \equiv \Delta_1, \quad \theta_{HH} - \theta_{HL} \equiv \Delta_2, \;\; \theta_{HH} - \theta_{LH} \equiv \Delta_3, \\ &\theta_{HL} - \theta_{LL} \equiv \Delta_4, \quad\; \theta_{LH} - \theta_{LL} \equiv \Delta_5, \;\; \theta_{LH} - \theta_{HL} \equiv \Delta_6, \\ &\theta_{HL} - \theta_{LH} \equiv \Delta_7 \end{aligned} \tag{3}$$

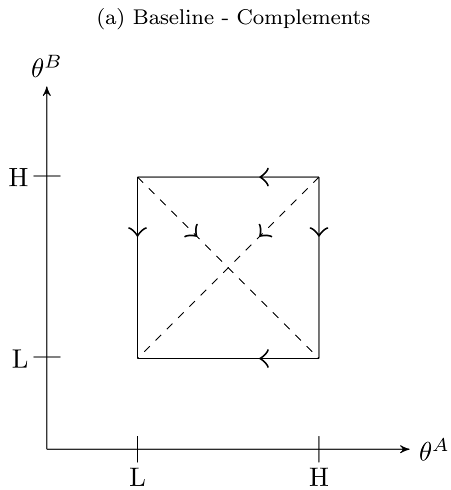
Only the downward IC constraints are incorporated into the maximization problem, just as in the case with unrelated products. The set of relevant incentive constraints is illustrated in figure 1. As with independent goods, there are four possible IC structures with complement goods. I will also refer to these as ICS $\Gamma$, $\Upsilon$, $\Psi$, $\Omega$.

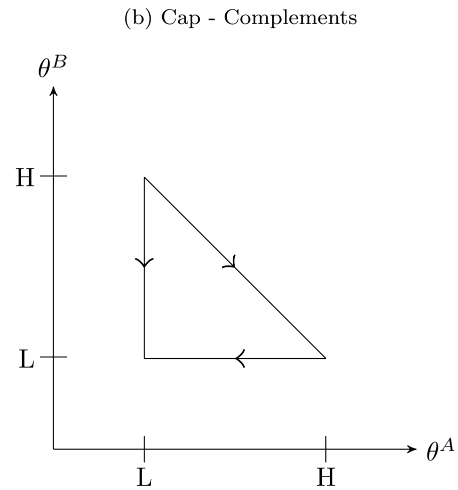
The set of first order conditions characterizing the solution to the seller's problem with related goods is shown in the appendix. Recall that in this model the goods are said to be bundled if the portion of item $i$ increases with good $j$. With complement goods, the seller does not bundle the products. Instead, the seller offers three packages with either only "large", only "medium" or only "small" portions of both goods each. The HL and LH-types are served the same option. Figure 2 serves as an example of the effects in the symmetric case.

Moving on to consumer surplus, without a cap, the HL and LL buyer types receive the same rents independently of the original ICS:

$$\begin{aligned} &R_{HL} = \bar{u} + \Delta_4[u(q_{LL}^A) + u(q_{LL}^B)] \\ &R_{LL} = \bar{u} \end{aligned} \tag{4}$$

Figure 1: Set of IC constraints




The rents earned by the LH-type in the regulation-free baseline vary in the following way:

$$\begin{aligned} &\text{ICS } \Gamma, \Upsilon, \Psi : R_{LH} = \Delta_5[u(q^A_{LL}) + u(q^B_{LL})] \\ &\text{ICS } \Omega : R_{LH} = (\Delta_5 + \Delta_4)[u(q^A_{LL}) + u(q^B_{LL})] + \Delta_6[u(q^A_{HL}) + u(q^B_{HL})] \end{aligned} \tag{5}$$

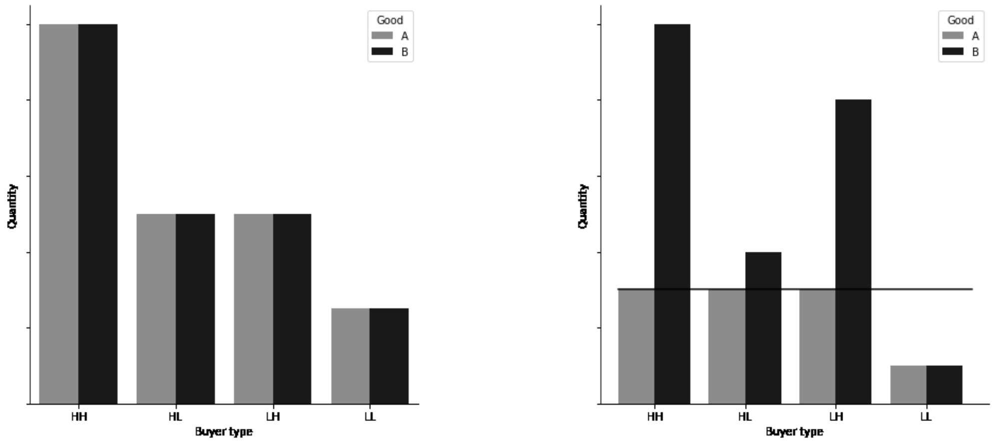
The information rents received by the HH-type depends on the IC structure as follows:

$$\begin{aligned} &\text{ICS } \Gamma : R_{HH} = (\Delta_4 + \Delta_5)[u(q^A_{LL}) + u(q^B_{LL})] + \Delta_3[u(q^A_{LH}) + u(q^B_{LH})] + \Delta_2[u(q^A_{HL}) + u(q^B_{HL})] \\ &\text{ICS } \Upsilon : R_{HH} = (\Delta_4 + \Delta_5)[u(q^A_{LL}) + u(q^B_{LL})] + \Delta_3[u(q^A_{LH}) + u(q^B_{LH})] + (\Delta_6 + \Delta_2)[u(q^A_{HL}) + u(q^B_{HL})] \\ &\text{ICS } \Psi R_{HH} = (\Delta_1 + \Delta_4 + \Delta_5)[u(q^A_{LL}) + u(q^B_{LL})] + \Delta_3[u(q^A_{LH}) + u(q^B_{LH})] + \Delta_1[u(q^A_{HL}) + u(q^B_{HL})] \\ &\text{ICS } \Omega R_{HH} = (\Delta_1 + \Delta_4 + \Delta_5)[u(q^A_{LL}) + u(q^B_{LL})] + \Delta_3[u(q^A_{LH}) + u(q^B_{LH})] + (\Delta_6 + \Delta_2)[u(q^A_{HL}) + u(q^B_{HL})] \end{aligned} \tag{6}$$

Regarding consumer surplus; the HH-type receives the most information rents; the LL-type the less, and between these types, the HL and LH buyers receive the same level of surplus.

## 1.1 Quantity cap with related goods

I continue to study the same three levels of cap severity on product A introduced in the section where I look at the case with independent goods. Recall that the mild cap directly limits only the




(c) Baseline - Related goods

(d) Cap - Related goods

Figure 2: Allocation by Buyer Types (Theory)

large portion of A, the moderate regulation directly affects both medium and large portions of A, while the severe limit directly affects all options of A. The seller's goal is to maximize expected profit subject to the relevant IC and PC restrictions plus the quantity cap.

**Proposition 1**: *Moderate and severe caps change the information rents earned by the HH and LH types to the following:*

- *Moderate cap:*
  $\widetilde{R_{HH}} = (\Delta_4 + \Delta_1)[u(\widetilde{q^A_{LL}}) + u(\widetilde{q^B_{LL}})] + \Delta_2[u(\bar{q}) + u(\widetilde{q^B_{HL}})]$
  $\widetilde{R_{LH}} = (\Delta_5 + \Delta_4)[u(\widetilde{q^A_{LL}}) + u(\widetilde{q^B_{LL}})] + \Delta_6[u(\bar{q}) + u(\widetilde{q^B_{HL}})]$

- *Severe cap:*
  $\widehat{R_{HH}} = (\Delta_4 + \Delta_1)[u(\bar{q}) + u(\widehat{q^B_{LL}})] + \Delta_2[u(\bar{q}) + u(\widehat{q^B_{HL}})]$
  $\widehat{R_{LH}} = (\Delta_5 + \Delta_4)[u(\bar{q}) + u(\widehat{q^B_{LL}})] + \Delta_6[u(\bar{q}) + u(\widehat{q^B_{HL}})]$

*Direct comparison with the corresponding information rents equations yield the following results. Following moderate and severe caps and compared to the rents earned without regulation:*

- *$R_{HH}$ is diminished.*

- *The effect on $R_{HL}$ is ambiguous and depends on the model's specific parameter values.*

- *$R_{LH}$ unambiguously increases for regulation free IC-structures $\Gamma$, $\Upsilon$, and $\Psi$. For IC-structure $\Omega$, $R_{LH}$ increases as long as:*
  *Moderate:*
  $(\Delta_5 + \Delta_4)[u(q^A_{LL}) - u(\widetilde{q^A_{LL}}) + u(q^B_{LL}) - u(\widetilde{q^B_{LL}})] + \Delta_6[u(q^A_{LL}) - u(\bar{q}) + u(q^B_{HL}) - u(\widetilde{q^B_{HL}})] < 0$
  *Severe:*
  $(\Delta_5 + \Delta_4)[u(\bar{q}) - u(\widehat{q^A_{LL}}) + u(q^B_{LL}) - u(\widehat{q^B_{LL}})] + \Delta_6[u(q^A_{LL}) - u(\bar{q}) + u(q^B_{HL}) - u(\widehat{q^B_{HL}})] < 0$

- *$R_{LL}$ remains unaffected.*

The consumer surplus granted to the LH-type are larger under regulation-free ICS$\Gamma$, $\Upsilon$, and $\Psi$. For ICS $\Omega$, the effect on LH is likely to be positive because $u(q^A_{LL}) - u(\widetilde{q^A_{LL}}) < 0$; $u(q^B_{LL}) - u(\widetilde{q^B_{LL}}) < 0$, and $u(q^B_{HL}) - u(\widetilde{q^B_{HL}}) < 0$; likewise $u(q^B_{LL}) - u(\widehat{q^B_{LL}}) < 0$, and $u(q^B_{HL}) - u(\widehat{q^B_{HL}}) < 0$.

**Proposition 2**: *A mild cap only affects $q^A_{HH}$. Following a moderate cap:*

- *$q^A_{HH}$, $q^A_{HL}$, and $q^A_{LH}$ are directly affected.*

- *$q^A_{LL}$ increases for original IC-Structure $\Omega$, and the effect is contingent to the parametrization for the rest of IC-structures.*

- *$q_{HH}$ does not change.*

- *$q^B_{HL}$ unambiguously decreases for ICS $\Gamma$, unambiguously increases for ICS $\Omega$, and depends on specific parameter values for the rest of IC-structures.*

- *$q^B_{LH}$ unambiguously increases for IC-structures $\Gamma$, $\Upsilon$, and $\Psi$. It decreases for ICS $\Omega$.*

- *$q^B_{LL}$ increases for the original ICS $\Omega$, and the effect depends on the specific parameter values for the rest of IC-structures*

These changes are straightforwardly corroborated by a simple comparison between the corresponding first order conditions. Table 1 summarizes the effect of the cap for each severity level comparing the resulting quantities to the quantities allocated to each type under no regulation. The first order conditions characterizing the solutions are in the appendix. Just as in the case with

Table 1: Theoretical change in quantities: Complement and substitute goods

| | $q^A_{HH}$ | $q^B_{HH}$ | $q^A_{HL}$ | $q^B_{HL}$ | $q^A_{LH}$ | $q^B_{LH}$ | $q^A_{LL}$ | $q^B_{LL}$ |
|---|---|---|---|---|---|---|---|---|
| **IC-Structure $\Gamma$** | | | | | | | | |
| Mild | $\downarrow$ | = | = | = | = | = | = | = |
| Moderate | $\downarrow$ | = | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\uparrow$ | $\downarrow^1$ | $\downarrow^1$ |
| Severe | $\downarrow$ | = | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\uparrow$ | $\downarrow$ | $\downarrow^1$ |
| **IC-Structure $\Upsilon$** | | | | | | | | |
| Mild | $\downarrow$ | = | = | = | = | = | = | = |
| Moderate | $\downarrow$ | = | $\downarrow$ | $\downarrow^2$ | $\downarrow$ | $\uparrow$ | $\downarrow^3$ | $\downarrow^1$ |
| Severe | $\downarrow$ | = | $\downarrow$ | $\downarrow^2$ | $\downarrow$ | $\uparrow$ | $\downarrow$ | $\downarrow^1$ |
| **IC-Structure $\Psi$** | | | | | | | | |
| Mild | $\downarrow$ | = | = | = | = | = | = | = |
| Moderate | $\downarrow$ | = | $\downarrow$ | $\downarrow^4$ | $\downarrow$ | $\uparrow$ | $\downarrow^5$ | $\downarrow^5$ |
| Severe | $\downarrow$ | = | $\downarrow$ | $\downarrow^4$ | $\downarrow$ | $\uparrow$ | $\downarrow$ | $\downarrow^5$ |
| **IC-Structure $\Omega$** | | | | | | | | |
| Mild | $\downarrow$ | = | = | = | = | = | = | = |
| Moderate | $\downarrow$ | = | $\downarrow$ | $\uparrow$ | $\downarrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ |
| Severe | $\downarrow$ | = | $\downarrow$ | $\uparrow$ | $\downarrow$ | $\uparrow$ | $\downarrow$ | $\uparrow$ |

In each case, the comparison is against the baseline scenario. For arrows with superscript, the effect holds if the following inequalities hold:
$^1\,\frac{\beta_{HH}}{\beta_{LH}} < \frac{\Delta_4}{\Delta_5-\Delta_1}$. $^2\,\beta_{HH} < \beta_{LH}$. $^3\,\Delta_5 < \Delta_1 + \Delta_4$. $^4\,\frac{\beta_{HH}}{\beta_{LH}} < \frac{\Delta_6}{\Delta_1-\Delta_2}$. $^5\,\frac{\beta_{HH}}{\beta_{LH}} < \frac{\Delta_4}{\Delta_5}$.

independent goods, the mild cap does not modify the quantities of the products beyond the largest alternatives.

Moderate and severe caps have more nuanced effects on quantities. I first discuss the effects on product A. By design, a moderate cap directly reduces the large and medium portions of good A. The moderate cap also indirectly affects the quantity of A contained in the small package. When under no regulation, the market is characterized by IC structure $\Gamma$, $\Upsilon$, or $\Psi$ a moderate cap can either reduce or increase the quantity of A contained in the smallest package, depending on the value of the model's parameters as shown in the relevant footnote in table 1. For ICS $\Omega$, where $R_{HH}$

takes the form of last equation in 6, a moderate restriction unambiguously increases the portion of A contained in the small option.

Moving to the impacts on the unregulated good B, with regulation-free ICS $\Gamma$, the quantity of good $\Upsilon$ served to the HL type unambiguously decreases; with ICS $\Psi$, this type is served a larger portion of B; while with IC structures $\Upsilon$ and $\Psi$ the effect is ambiguous and might decrease contingent on particular parameter values. The quantity of product B consumed by buyer type LH increases regardless of the original ICS. While the portion served to buyer type LL unambiguously increases for ICS $\Omega$, and may decrease for the rest of IC structures depending on parameter values.

For an interpretation of the results observed with related goods, consider seller of SSBs deciding sugar-"water" combinations (possible A-B products, where "water" is a composite good that includes ingredients such as flavoring). Suppose that the context is such that we observe allocations resembling panels (c) and (d) in figure 2. The "package" is a bottle of soda with a particular sugar-water ratio. In this context, the baseline outcomes can be interpreted as follows. Without regulation, the seller decides to produce bottles of soda in three different presentations: small, medium, and large servings all with a one-to-one sugar-water ratio. If the government enacts a limit on the maximum amount grams of sugar contained in a single serving, the seller would accommodate the policy by offering the following four choices. First, a "light" large alternative with low sugar-water ratio serving the HH-type (who, after all, also highly values the ingredients other than sugar contained in the beverage); second, a relatively small option with a concentrated formula with a high sugar-water ratio designed for the HL-type's sweet taste; third, a smaller "light" alternative serving the health-conscious LH-type; and lastly, a mini serving of the "traditional" formula targeting the LL-type buyer.

## Proof of proposition 1

The analysis follows closely what I show in the proof of proposition

$$\begin{aligned}
LL \to LH: &\quad R_{LL} \geq R_{LH} - \Delta_5 u(\bar{q}) - \Delta_5 u(q^B_{LH}) \\
LL \to HL: &\quad R_{LL} \geq R_{HL} - \Delta_4 u(\bar{q}) - \Delta_4 u(q^B_{HL}) \\
LL \to HH: &\quad R_{LL} \geq R_{HH} - \Delta_1 u(\bar{q}) - \Delta_1 u(q^B_{HH}) \\
LH \to LL: &\quad R_{LH} \geq R_{LL} + \Delta_5 u(q^A_{LL}) + \Delta_5 u(q^B_{LL}) \\
LH \to HL: &\quad R_{LH} \geq R_{HL} + \Delta_6 u(\bar{q}) + \Delta_6 u(q^B_{HL}) \\
LH \to HH: &\quad R_{LH} \geq R_{HH} - \Delta_3 u(\bar{q}) - \Delta_3 u(q^B_{HH}) \\
HL \to LL: &\quad R_{HL} \geq R_{LL} + \Delta_4 u(q^A_{LL}) + \Delta_4 u(q^B_{LL}) \\
HL \to LH: &\quad R_{HL} \geq R_{LH} + \Delta_7 u(\bar{q}) + \Delta_7 u(q^B_{LH}) \\
HL \to HH: &\quad R_{HL} \geq R_{HH} - \Delta_2 u(\bar{q}) - \Delta_2 u(q^B_{HH}) \\
HH \to LL: &\quad R_{HH} \geq R_{LL} + \Delta_1 u(q^A_{LL}) + \Delta_1 u(q^B_{LL}) \\
HH \to LH: &\quad R_{HH} \geq R_{LH} + \Delta_3 u(\bar{q}) + \Delta_3 u(q^B_{LH}) \\
HH \to HL: &\quad R_{HH} \geq R_{HL} + \Delta_2 u(\bar{q}) + \Delta_2 u(q^B_{HL})
\end{aligned} \tag{7}$$

From the possible candidate combinations of forms of $R_{HL}$ and $R_{LH}$, the following do not violate any of the IC constraints listed above:

$$\begin{aligned}
\widetilde{R_{LH}} &= (\Delta_5 + \Delta_4)[u(\widetilde{q^A_{LL}}) + u(\widetilde{q^B_{LL}})] + \Delta_6[u(\bar{q}) + u(\widetilde{q^B_{HL}})] \\
\widetilde{R_{HL}} &= \Delta_4[u(\widetilde{q^A_{LL}}) + u(\widetilde{q^B_{LL}})]
\end{aligned} \tag{8}$$